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On Equivalent Definitions of the Correlation Dimension for a Probability Measure

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In mathematical physics, one sometimes has to deal with averages of the type

$$M\mu(T)=\frac{1}{T^n}\int_{|\xi|\leqslant T} d\xi\, |\hat{\mu}(\xi)|^2, \qquad T>0$$

where $\hat{\mu}$ is the Fourier transform of some probability Borel measure μ. We show that the asymptotic behavior of $M\mu$ is governed by the usual (upper and lower) correlation dimension of the measure μ.

KEY WORDS: Fractal measure; correlation dimension.

1. INTRODUCTION

In the past decade the study of quantum systems in connection with the fractal formalism has become more and more of interest. There are at least two typical situations where one has to deal with singular measures, that is, measures with nowhere-dense support. The first one is the problem of localization for systems described by Hamiltonians with singular continuous spectra, such as the quasiperiodic or disordered quantum systems. The second one is the scattering on nonsmooth obstacles. To be more concrete, consider the following two characteristic examples, referred to as Problem A and Problem B, respectively.

Problem A. Localization of States. Let H be a self-adjoint operator acting on some Hilbert space $\mathscr{H}$. The dynamics of a quantum system with Hamiltonian H is determined by the one-parameter continuous

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group of unitary operators $\mathscr{U}(t) = e^{-iHt}$. If ϕ_0 in $\mathscr{H}$ is the initial state of the system, the time evolution is given by

$$\phi_t = \mathscr{U}(t)\,\phi_0$$

One is interested in the diffusive properties of the quantum system. This is mirrored in the time-space behavior of the wavepackets ϕ_t. The evolving state ϕ_t may propagate away from its initial localization or on the contrary stay confined in some finite region. The rate of delocalization can be characterized by the time averages

$$M(T) = \frac{1}{T}\int_0^T |\langle \phi_t | \phi_0 \rangle|^2\, dt$$

If we introduce μ, the spectral measure of H associated to ϕ_0, then, by the usual functional calculus (see, e.g. ref. 6), we have

$$M(T) = \frac{1}{T}\int_0^T |\hat{\mu}(t)|^2\, dt$$

where $\hat{\mu}$ is the Fourier transform of μ:

$$\hat{\mu}(t) = \int dx\, e^{-ixt}\, d\mu(x)$$

Now, the asymptotic behavior of $M(T)$ is connected to the degree of singularity of the spectral measure. If μ is an atomic measure, then a well-known theorem due to Wiener states that

$$M(T) \sim 1, \qquad T \to \infty$$

In this case no diffusion is possible and the states stay localized in configuration space. If on the contrary μ is absolutely continuous with some density function $f \in \mathscr{L}^2(\mathbb{R})$ one has the Plancherel identity $\int d\xi\, |\hat{\mu}(\xi)|^2 = \int dx\, |f(x)|^2$, whence

$$M(T) \sim T^{-1}, \qquad T \to \infty$$

implying a complete diffusion of the wave packets. In the case of a singular continuous measure μ one might expect an intermediate behavior. Some heuristic arguments have been given in ref. 4 supporting the fact that the average $M(T)$ exhibits a scaling behavior:

$$M(T) \sim T^{-D}$$

where D is the correlation dimension of the measure μ. In ref. 3 some new fractal dimensions have been introduced by means of wavelet transforms. For these dimensions it has been shown that actually the heuristic argument is true and that the long-time evolution of the averages $M(T)$ is governed by the upper, respectively lower, wavelet correlation dimension.

Now, the same type of average arises in another situation:

Problem B. One-Dimensional Potential Scattering. Consider the one-dimensional stationary Schrödinger equation

$$\left(\frac{d^2}{dx^2}+p^2\right)\Psi_{V,p}=V\Psi_{V,p} \tag{1.1}$$

where V is a multiplication operator by some smooth real-valued function $V(x)$ with compact support. Left of the support of V the solution $\Psi_{V,p}$ of (1.1) has the form $\Psi_{V,p}(x)=Ae^{ipx}+Be^{-ipx}$. Right of the support it has the same form with other coefficients, say $\Psi_{V,p}(x)=A'e^{ipx}+B'e^{-ipx}$. For each p there is a unique pair of complex numbers $t(p)$ and $r(p)$ such that

$$\Psi_{V,p}=\begin{cases} e^{ipx}+r(p)\,e^{-ipx}, & x\to-\infty \\ t(p)\,e^{ipx}, & x\to+\infty \end{cases}$$

The coefficients $t(p)$ and $r(p)$ are known, respectively, as the transmission and reflection amplitudes on the potential barrier V. It can be shown that the reflection coefficient has the following asymptotic behavior as $p\to\infty$:

$$r(p)=\frac{\hat{V}(-2p)}{2ip}+O\left(\frac{1}{p^2}\right) \tag{1.2}$$

where $\hat{V}(\xi)=\int dx\, e^{-i\xi x}V(x)$ is the usual Fourier transform. Now suppose that we want to deal with a potential of very poor regularity (for example, a point interaction). In this case the potential $V(x)$ can no longer be considered as a smooth function; therefore we assume that it is merely a measure. As shown in ref. 2, the scattering problem defined for smooth potentials can actually be extended to any finite Borel measure with compact support. In particular, the asymptotic (1.2) is still valid, $\hat{V}$ being now understood as the Fourier transform of the Borel measure V.

One might expect that the high-energy asymptotic of the scattering data reveals the small-scale structure of the scatterer and thus it would be natural to observe some scaling behavior $r(p)\sim p^{-\alpha}$, $p\to\infty$. However, this is not the case in general because the reflection coefficient can be very chaotic. Therefore, one rather looks at the averages

$$\int_0^P dp\, p^2\, |r(p)|^2 \sim \int_0^P dp\, |\hat{V}(p)|^2, \qquad P \to \infty$$

which are known to scale with the wavelet correlation dimension. This kind of average on the scattering data was first used in ref. 1 in the case of optical diffraction, where the authors recovered the fractal dimension of a Cantor-like grating illuminated by a spherical wave.

At this point, let us summarize the problem we want to consider. If we generalize the situation to the n-dimensional case, we are interested in the asymptotic behavior of

$$M\mu(T) = \frac{1}{T^n} \int_{|\xi| \leqslant T} d\xi\, |\hat{\mu}(\xi)|^2$$

where $\hat{\mu}$ is the Fourier transform of some finite Borel measure μ on $\mathbb{R}^n$. As outlined in the last two examples, these averages are governed by some fractal dimension of the measure μ, namely the wavelet correlation dimension. However, this latter dimension is rather complicated and furthermore is not well known. Therefore we wish to establish a correspondence with a more common fractal dimension. We define two exponents

$$m^+(\mu) = \limsup_{T \to +\infty} \frac{\log M\mu(T)}{\log T}, \qquad m^-(\mu) = \liminf_{T \to +\infty} \frac{\log M\mu(T)}{\log T}$$

In this paper we show that $m^+(\mu)$ and $m^-(\mu)$ coincide with the usual upper and lower correlation dimensions in the case of probability Borel measures. We believe this equivalence to be known (at least partially) by specialists, although we have not found it in the literature. Our aim is here to clarify this subject.

2. STATEMENT OF THE PROBLEM AND FORMULATION OF MAIN RESULT

Consider a probability Borel measure μ on $\mathbb{R}^n$. One common definition[5] of its correlation dimension is the following. Take $B(x, a)$ as the closed ball of radius a around x in $\mathbb{R}^n$ and consider the function $x \mapsto \mu(B(x, a))$. This is a μ-measurable function. Indeed, if we decompose μ

into a purely direct part, say $\mu_{pp} = \sum a_i \delta_{b_i}$, and a continuous part, say μ_c, we have that for all $x \in \mathbb{R}^n$,

$$\mu(B(x, a)) = \sum a_i \chi_{B(b_i, a)}(x) + \mu_c(B(x, a))$$

where the notation χ_A designates the characteristic function of a set $A \subset \mathbb{R}^n$. The first term is measurable; the second one is a continuous function. Hence, we can form the μ-average:

$$\Omega\mu(a) = \int d\mu(x)\, \mu(B(x, a))$$

The quantities

$$\omega^+(\mu) = \limsup_{a \to 0} \frac{\log \Omega\mu(a)}{\log a}$$

$$\omega^-(\mu) = \liminf_{a \to 0} \frac{\log \Omega\mu(a)}{\log a}$$

are called, respectively, the upper and lower correlation dimensions of μ.

Now, this definition of the correlation dimension is not very handy, because the computation of $\Omega\mu$ involves an integration versus $d\mu$. In particular, this forbids numerical applications. However, if we formally replace $\mu(x)$ by its mean value $a^{-n}\mu(B(x, a))$ on a ball of radius a, we obtain a slightly different version of $\Omega\mu$:

$$\Gamma\mu(a) = \frac{1}{a^n} \int dx\, [\mu(B(x, a))]^2$$

where now the integration is versus dx, the Lebesgue measure. Note that we have implicitly supposed $x \mapsto \mu(B(x, a))$ to be square-integrable. This is actually the case, as we shall prove in the sequel. We also introduce the function

$$M\mu(T) = \frac{1}{T^n} \int_{|\xi| \leqslant T} d\xi\, |\hat{\mu}(\xi)|^2$$

where $\hat{\mu}$ is the Fourier transform of μ:

$$\hat{\mu}(\xi) = \int_{\mathbb{R}^n} d\mu(x)\, e^{-i\xi x} \tag{2.1}$$

The dimensions associated to $\Gamma\mu$ and $M\mu$ are, respectively,

$$\gamma^+(\mu) = \limsup_{a \to 0} \frac{\log \Gamma\mu(a)}{\log a}, \qquad \gamma^-(\mu) = \liminf_{a \to 0} \frac{\log \Gamma\mu(a)}{\log a}$$

and

$$m^+(\mu) = \limsup_{T \to +\infty} \frac{\log M\mu(T)}{\log T}, \qquad m^-(\mu) = \liminf_{T \to +\infty} \frac{\log M\mu(T)}{\log T}$$

In ref. 3 the dimensions $m^\pm(\mu)$ are shown to coincide, up to the sign, with the wavelet correlation dimensions. In this note we establish simple relations between the three pairs of dimensions $\omega^\pm(\mu)$, $\gamma^\pm(\mu)$, and $m^\pm(\mu)$. We will prove that

$$\gamma^+(\mu) = \omega^+(\mu) = -m^-(\mu)$$

$$\gamma^-(\mu) = \omega^-(\mu) = -m^+(\mu)$$

and thus show that the correlation dimension of a probability Borel measure can be defined via any one of the three functions $\Omega\mu$, $\Gamma\mu$, or $M\mu$.

3. SOME GENERALIZATIONS

Let χ be the characteristic function of the unit ball $B(0, 1)$ in $\mathbb{R}^n$, and χ_a, its dilated version:

$$\chi_a(x) = \frac{1}{a^n} \chi\left(\frac{x}{a}\right)$$

The functions $\Omega\mu$, $\Gamma\mu$, and $M\mu$ can be more simply rewritten as

$$\Omega\mu(a) = a^n \int d\mu(x)\, \mu * \chi_a(x)$$

$$\Gamma\mu(a) = a^n \int dx\, [\mu * \chi_a(x)]^2$$

$$M\mu(T) = \int d\xi\, |\hat{\mu}(\xi)|^2\, \chi_T(\xi)$$

A natural extension of these definitions is to replace χ by some positive function g well localized around zero. More precisely, let g be a positive Borel function, that is, a function for which all preimages $g^{-1}(A)$ of Borel sets $A \subset \mathbb{R}^n$ are Borel sets, too, and as before let $g_a(x) = a^{-n} g(x/a)$ be its dilation. Suppose in addition that g is rapidly decreasing, that is, it decays faster than every polynomial:

$$\max_{|\alpha| \leqslant N} \sup_{x \in \mathbb{R}^n} |x^\alpha g(x)| < \infty, \qquad N = 0, 1, 2, \ldots$$

Then it makes sense to replace χ by g, as shown in the following proposition.

Proposition 1. If g is a rapidly decreasing function, then for $a > 0$ the functions

$$\Omega_g \mu(a) = a^n \int d\mu(x)\, \mu * g_a(x) \tag{3.1}$$

$$\Gamma_g \mu(a) = a^n \int dx\, (\mu * g_a(x))^2 \tag{3.2}$$

$$M_g \mu(T) = \int d\xi\, |\hat{\mu}(\xi)|^2\, g_T(\xi) \tag{3.3}$$

are well defined and take finite values. If, furthermore, g belongs to the Schwartz class $\mathscr{S}(\mathbb{R}^n)$, then the Parseval–Bessel equality for $\Omega_g \mu$ and $\Gamma_g \mu$ holds, namely

$$\Omega_g \mu(a) = \frac{a^n}{(2\pi)^n} \int d\xi\, |\hat{\mu}(\xi)|^2\, \hat{g}(a\xi) \tag{3.4}$$

and

$$\Gamma_g \mu(a) = \frac{a^n}{(2\pi)^n} \int d\xi\, |\hat{\mu}(\xi)|^2\, |\hat{g}(a\xi)|^2 \tag{3.5}$$

Note that the average in (3.5) coincides with the one given in ref. 3 in relation with the wavelet correlation dimension.

Proof. The proofs are elementary, but we shall give them anyway for the convenience of the reader.

Proof of (3.1). Since g_a is measurable, there exists a sequence of step functions g_a^N converging pointwise to g_a as $N \to \infty$ (see, e.g., ref. 7). By the same argument as above each function $\mu * g_a^N$ is measurable. Hence the limit function $\mu * g_a$ is also measurable. Moreover, g_a is bounded by $a^{-n} \|g\|_\infty$ and

$$\Omega_g \mu(a) \leqslant \|g\|_\infty \int d\mu(x)\, \mu * 1(x) = \|g\|_\infty$$

Proof of (3.2). The function g together with its dilations g_a are in $\mathscr{L}^1$ and by Fubini's theorem the same holds for $\mu * g_a$. Now clearly $\mu * g_a$ is in $\mathscr{L}^\infty$ and thus $\mu * g_a$ is in $\mathscr{L}^2$ by Hölder's inequality.

Proof of (3.3). The Fourier transform of a probability measure is a continuous function and there is no problem in defining $M_g\mu$. Moreover, $|\hat{\mu}|^2$ is bounded by one and $|M_g\mu(T)| \leqslant \|g\|_1$.

Proof of (3.4). Choose a nonnegative function ϕ in C_0^∞ such that $\phi = 1$ on the unit ball $B(0, 1)$ and form the sequence of functions

$$\Omega_g^N \mu(a) = a^n \int d\mu(x)\, \mu * g_a(x)\, \phi\left(\frac{x}{N}\right)$$

These functions converge for each a to $\Omega_g\mu(a)$ since

$$\begin{aligned}|\Omega_g^N \mu(a) - \Omega_g\mu(a)| &\leqslant a^n \int d\mu(x)\, |\mu * g_a(x)| \cdot \left|\phi\left(\frac{x}{N}\right) - 1\right| \\ &\leqslant O(1)\, a^n \int_{|x| \geqslant N} d\mu(x) \to 0, \qquad N \to \infty\end{aligned}$$

Since we may, we take μ as a tempered distribution and hence by definition we have

$$\Omega_g^N \mu(a) = \frac{a^n}{(2\pi)^n} \int d\xi\, \bar{\hat{\mu}}(\xi) \left[\mu * g_a(\cdot)\, \phi\left(\frac{\cdot}{N}\right)\right]^\wedge (\xi)$$

Here $\hat{\mu}$ is *a priori* taken in the sense of distributions, but since $\hat{\mu}$ is a continuous function, it coincides with Definition 2.1. Now for $\eta \in \mathscr{S}'$ and $s \in \mathscr{S}$ the convolution theorem applies, namely $(\eta * s)^\wedge = \hat{\eta} \cdot \hat{s}$, and therefore we have

$$\left[\mu * g_a(\cdot)\, \phi\left(\frac{\cdot}{N}\right)\right]^\wedge (\xi) = (\widehat{\hat{\mu} g_a}) * \hat{\phi}_{1/n}(\xi)$$

Altogether we end up with

$$\Omega_g^N \mu(a) = \frac{a^n}{(2\pi)^n} \int d\xi\, \bar{\hat{\mu}}(\xi) (\widehat{\hat{\mu} g_a}) * \hat{\phi}_{1/N}(\xi)$$

Since $\widehat{\hat{\mu} g_a}$ is continuous and $\hat{\phi}$ rapidly decaying, it follows from the approximation of the identity lemma that $(\widehat{\hat{\mu} g_a}) * \hat{\phi}_{1/N}$ converges pointwise toward $\widehat{\mu g_a}$. Moreover, $(\widehat{\mu g_a}) * \hat{\phi}_{1/N}$ is uniformly bounded by some $\mathscr{L}^1$ function, as shown in the following lemma:

Lemma 3.1. Consider the function

$$K^\alpha(x) = (1 + |x|)^{-\alpha}$$

Then for all $\alpha > n$ we have

$$|K^\alpha_{1/N} * K^\alpha(x)| \leqslant O(1)\, K^\alpha(x)$$

uniformly in $N \geqslant 1$, where as usual $K_{1/N}(\cdot) = NK(N\cdot)$.

Proof. For any $x \in \mathbb{R}^n$ we split the convolution integral into two domains:

$$\begin{aligned} K^\alpha_{1/N} * K^\alpha(x) &= \left\{ \int_{y \in B(0,\, |x/2|)} + \int_{y \notin B(0,\, |x/2|)} \right\} dy\, K^\alpha_{1/N}(x-y)\, K^\alpha(y) \\ &= I_1(x) + I_2(x) \end{aligned}$$

For all $y \in B(0, |x/2|)$, we have $|x - y| \geqslant |x|/2$. Hence

$$I_1(x) \leqslant K^\alpha_{1/N}\left(\frac{x}{2}\right) \int_{y \in B(0,\, |x/2|)} dy\, |K^\alpha(y)| \leqslant \|K^\alpha\|_1\, K^\alpha_{1/N}\left(\frac{x}{2}\right)$$

On the other hand,

$$I_2(x) \leqslant K^\alpha\left(\frac{x}{2}\right) \int_{y \notin B(0,\, |x/2|)} dy\, |K^\alpha_{1/N}(x-y) \leqslant K^\alpha\left(\frac{x}{2}\right) \|K^\alpha\|_1$$

Now on every domain bounded away from zero, say $\mathbb{R}^n \backslash B(0, 1)$, we have

$$K^\alpha_{1/N}(x) \leqslant O(|x|^{-\alpha})\, O(N^{n-\alpha}) \leqslant O(1)\, K^\alpha(x/2)$$

and thus

$$K^\alpha_{1/N} * K^\alpha = I_1(x) + I_2(x) \leqslant O(1)\, K^\alpha(x/2) \leqslant O(1)\, K^\alpha(x) \tag{3.6}$$

Since $K^\alpha_{1/N} * K^\alpha$ is uniformly bounded by $\|K^\alpha\|_1\, \|K^\alpha\|_\infty$ on $B(0, 1)$, (3.6) holds on the whole space $\mathbb{R}^n$ and the proof of Lemma 3.1 is completed. ∎

As a result, we have that

$$|(\widehat{\mu g_a}) * \hat{\phi}_{1/N}| \leqslant O(1)\, K^\alpha \in \mathscr{L}^1(\mathbb{R}^n)$$

for all $\alpha > n$ uniformly in $N \geqslant 1$. We use the Lebesgue dominated convergence theorem to conclude

$$\Omega_g \mu(a) = \lim_{N \to \infty} \Omega_g^N \mu(a) = a^n \int d\xi \, |\hat{\mu}(\xi)|^2 \, \hat{g}(a\xi)$$

Proof of (3.5). This is similar and we leave it to the reader. ∎

Thus we have defined more general functions $\Omega_g \mu$, $\Gamma_g \mu$, and $M_g \mu$ parametrized by g. We can associate to them a set of exponents that are a natural generalization of the previous exponents:

$$\omega_g^+(\mu) = \limsup_{a \to 0} \frac{\log \Omega_g \mu(a)}{\log a}, \qquad \omega_g^-(\mu) = \liminf_{a \to 0} \frac{\log \Omega_g \mu(a)}{\log a}$$

$$\gamma_g^+(\mu) = \limsup_{a \to 0} \frac{\log \Gamma_g \mu(a)}{\log a}, \qquad \gamma_g^-(\mu) = \liminf_{a \to 0} \frac{\log \Gamma_g \mu(a)}{\log a}$$

$$m_g^+(\mu) = \limsup_{T \to +\infty} \frac{\log M_g \mu(T)}{\log T}, \qquad m_g^-(\mu) = \liminf_{T \to +\infty} \frac{\log M_g \mu(T)}{\log T}$$

4. MAIN RESULT

The main result we want to prove can be summarized in the following theorem.

Theorem 1. Let μ be a probability Borel measure on $\mathbb{R}^n$ and g a Borel function on $\mathbb{R}^n$. If g is nonnegative, rapidly decreasing, continuous at zero, and such that $g(0) > 0$, then we have

$$m_g^+(\mu) = m^+(\mu), \qquad m_g^-(\mu) = m^-(\mu) \tag{4.1}$$

$$\gamma_g^+(\mu) = -m^-(\mu), \qquad \gamma_g^-(\mu) = -m^+(\mu) \tag{4.2}$$

$$\omega_g^+(\mu) = -m^-(\mu), \qquad \omega_g^-(\mu) = -m^+(\mu) \tag{4.3}$$

Upon choosing g the characteristic function of $B(0, 1)$, we have the following result:

Corollary 1. The following holds:

$$\gamma^+(\mu) = \omega^+(\mu) = -m^-(\mu)$$

$$\gamma^-(\mu) = \omega^-(\mu) = -m^+(\mu)$$

For the proof of Theorem 1, we need the following lemma:

Lemma 1. Let $s(t)$ be a strictly positive function defined for $t>0$. Then we have

$$\liminf_{t\to 0}\frac{\log s(t)}{\log t}=\sup\{\alpha\,|\,s(t)\leqslant O(t^{\alpha}),\ t\to 0\}$$

$$\limsup_{t\to 0}\frac{\log s(t)}{\log t}=\inf\{\alpha\,|\,t^{\alpha}\leqslant O(s(t)),\ t\to 0\}$$

The proof of this lemma can be found e.g. in ref. 3.

Proof. We successively are going to prove Eqs. (4.1)–(4.3).

Proof of (4.1). Since $g(0)>0$ and g is continuous at zero, it is greater than some characteristic function, that is, we can find $\lambda>0$ such that

$$\frac{1}{\lambda}\chi(\lambda x)\leqslant g(x)$$

in which case

$$\chi_T(x)\leqslant\lambda^{n+1}g_{\lambda T}(x)$$

and

$$M\mu(T)\leqslant\lambda^{n+1}M_g\mu(\lambda T)\tag{4.4}$$

Now let us fix $\varepsilon>0$ and split $M_g\mu$ into two terms:

$$M_g\mu(T)=\left\{\int_{|\xi|\leqslant T^{1+\varepsilon}}+\int_{|\xi|>T^{1+\varepsilon}}\right\}d\xi\ |\hat{\mu}(\xi)|^2\ g_T(\xi)$$

The first part can be related to $M\mu$:

$$\int_{|\xi|\leqslant T^{1+\varepsilon}}d\xi\ |\hat{\mu}(\xi)|^2\ g_T(\xi)\leqslant\|g\|_{\infty}\ T^{\varepsilon}M\mu(T^{1+\varepsilon})$$

and the queue of the integral can be easily bounded:

$$\left|\int_{|\xi|>T^{1+\varepsilon}}d\xi\ |\hat{\mu}(\xi)|^2\ g_T(\xi)\right|\leqslant\int_{|\xi|>T^{\varepsilon}}d\xi\ |g(\xi)|$$

On one hand $T^{\varepsilon} M\mu(T^{1+\varepsilon})$ is a positive function which decreases not faster than $T^{\varepsilon-1}$ [this is an evident consequence of $\hat{\mu}(0) > 0$ and $\hat{\mu}$ continuous]. On the other hand, $\int_{|\xi| > T^{\varepsilon}} d\xi \, |g(\xi)|$ is a rapidly decaying function of T. Hence, we can regroup the two terms under the same bound:

$$M_g \mu(T) \leqslant O(1) \, T^{\varepsilon} M\mu(T^{1+\varepsilon}) \tag{4.5}$$

Inequalities (4.4) and (4.5) imply the following relations for any real α:

$$M_g \mu(T) \leqslant O(1) \, T^{\alpha} \Rightarrow M\mu(T) \leqslant O(1) \, T^{\alpha}$$

$$T^{\alpha} \leqslant O(1) \, M\mu(T) \Rightarrow T^{\alpha} \leqslant O(1) \, M_g \mu(T)$$

$$M\mu(T) \leqslant O(1) \, T^{\alpha} \Rightarrow M_g \mu(T) \leqslant O(1) \, T^{\alpha(1+\varepsilon)+\varepsilon}$$

$$T^{\alpha} \leqslant O(1) \, M_g \mu(T) \Rightarrow T^{\alpha} \leqslant T^{\varepsilon} O(1) \, M\mu(T^{1+\varepsilon})$$

Thanks to Lemma 1, this leads to

$$(1+\varepsilon)^{-1} \, m_g^-(\mu) \leqslant m^-(\mu) \leqslant m_g^-(\mu)$$

$$m_g^+(\mu) \leqslant m^+(\mu) \leqslant (1+\varepsilon) \, m_g^+(\mu) + \varepsilon$$

Since this holds for all $\varepsilon > 0$, it proves (4.1).

Proof of (4.2). The idea of the proof here is to bracket g by two Schwartz functions φ_1 and φ_2 and to use expressions (3.5) for $\Gamma_{\varphi_1}\mu$ and $\Gamma_{\varphi_2}\mu$.

Since $g(0) > 0$ and g is continuous at zero, there exists a positive function φ_1 in C_0^{∞} such that $\varphi_1 \leqslant g$ with $\varphi_1(0) > 0$. On the other hand, we can find a Schwartz function φ_2 which majorizes g. Therefore we have

$$\Gamma_{\varphi_1}\mu(a) \leqslant \Gamma_g \mu(a) \leqslant \Gamma_{\varphi_2}\mu(a)$$

with φ_i, $i = 1, 2$, $\in \mathcal{S}(\mathbb{R}^n)$. Now, taking (3.5) into account, we obtain

$$a^n \int d\xi \, |\hat{\mu}(\xi)|^2 \, |\hat{\varphi}_1(a\xi)|^2 \leqslant \Gamma_g \mu(a) \leqslant a^n \int d\xi \, |\hat{\mu}(\xi)|^2 \, |\hat{\varphi}_2(a\xi)|^2$$

that is,

$$M_{|\hat{\varphi}_1|^2} \mu\left(\frac{1}{a}\right) \leqslant \Gamma_g \mu(a) \leqslant M_{|\hat{\varphi}_2|^2} \mu\left(\frac{1}{a}\right)$$

The inequality is conserved when one takes the logarithm. Hence,

$$-m^{-}_{|\hat{\varphi}_1|^2}(\mu) \leqslant \omega^{+}_{g}(\mu) \leqslant -m^{-}_{|\hat{\varphi}_2|^2}(\mu)$$
$$-m^{+}_{|\hat{\varphi}_1|^2}(\mu) \leqslant \omega^{-}_{g}(\mu) \leqslant -m^{+}_{|\hat{\varphi}_2|^2}(\mu)$$

Now, since $\hat{\varphi}_i(0) = \int \varphi_i > 0$, the two functions $|\hat{\varphi}_i|^2$ fullfill the conditions of the first part of Theorem 1, whose direct application then proves (4.2). ▌

Proof of (4.3). The proof of (4.3) is similar to (4.2), but there is one more difficulty. Suppose we still have $\varphi_1 \leqslant g \leqslant \varphi_2$ with φ_i positive Schwartz functions. If we use expression (3.4) of Ω_{φ_i}, we have

$$a^n \int d\xi \, |\hat{\mu}(\xi)|^2 \, \bar{\hat{\varphi}}_1(a\xi) \leqslant \Omega_g \mu(a) \leqslant a^n \int d\xi \, |\hat{\mu}(\xi)|^2 \, \bar{\hat{\varphi}}_2(a\xi)$$

but the $\bar{\hat{\varphi}}_i$ are not necessarily positive real functions. Therefore, we must choose the φ_i more carefully. Assume φ_1 to be radial (this choice is always possible). Then the autoconvolution $\varphi_1 * \varphi_1$ is a C_0^∞ function whose Fourier transform takes real nonnegative values. Moreover, since $g(0) > 0$, we still can find some constant $c > 0$ such that

$$\frac{1}{c} \varphi_1 * \varphi_1(cx) \leqslant g(x)$$

On the other hand, we can majorize $|\hat{\varphi}_2|$ by some positive Schwartz function ϕ_2. Thus, setting $\phi_1(x) = (1/c^2)\, \hat{\varphi}_1^2(x/c)$, we have the new bracketing

$$a^n \int d\xi \, |\hat{\mu}(\xi)|^2 \, \phi_1(a\xi) \leqslant \Omega_g \mu(a) \leqslant a^n \int d\xi \, |\hat{\mu}(\xi)|^2 \, \phi_2(a\xi)$$

with real positive functions ϕ_1 and ϕ_2 and we can conclude the proof as before. ▌

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